Problem 1

Establish the commutative, distributive, and associative properties of the convolution integral.

(a) $f * g = g * f$

(b) $f * (g_1 + g_2) = f * g_1 + f * g_2$

(c) $f * (g * h) = (f * g) * h$

Solution

Part (a)

Start by using the definition of the convolution integral $f * g$.

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

Make the substitution $\xi = t - \tau$. Then $d\xi = -d\tau$.

$$f * g = \int_t^0 f(\xi)g(t-\xi)\,(-d\xi)$$

Use the minus sign to flip the limits of integration.

$$\begin{aligned} f * g &= \int_0^t f(\xi)g(t-\xi)\,d\xi \\ &= \int_0^t g(t-\xi)f(\xi)\,d\xi \\ &= g * f \end{aligned}$$

Part (b)

Start by using the definition of the convolution integral $f * (g_1 + g_2)$.

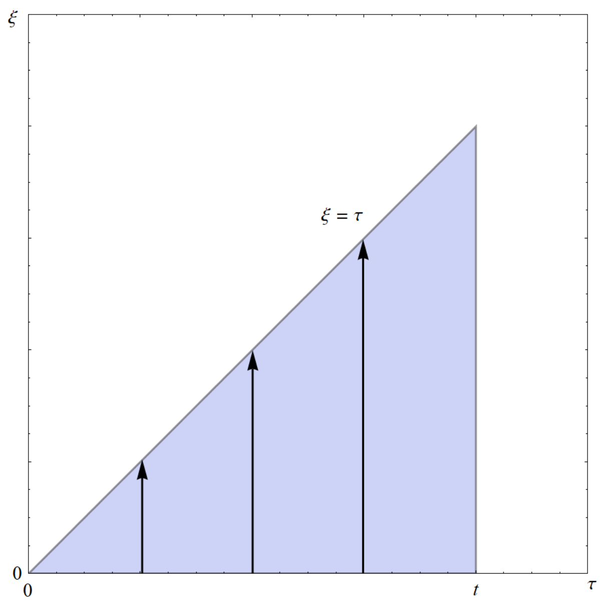
$$\begin{aligned} f * g &= \int_0^t f(t-\tau)[g_1(\tau) + g_2(\tau)]\,d\tau \\ &= \int_0^t [f(t-\tau)g_1(\tau) + f(t-\tau)g_2(\tau)]\,d\tau \\ &= \int_0^t f(t-\tau)g_1(\tau)\,d\tau + \int_0^t f(t-\tau)g_2(\tau)\,d\tau \\ &= f * g_1 + f * g_2 \end{aligned}$$

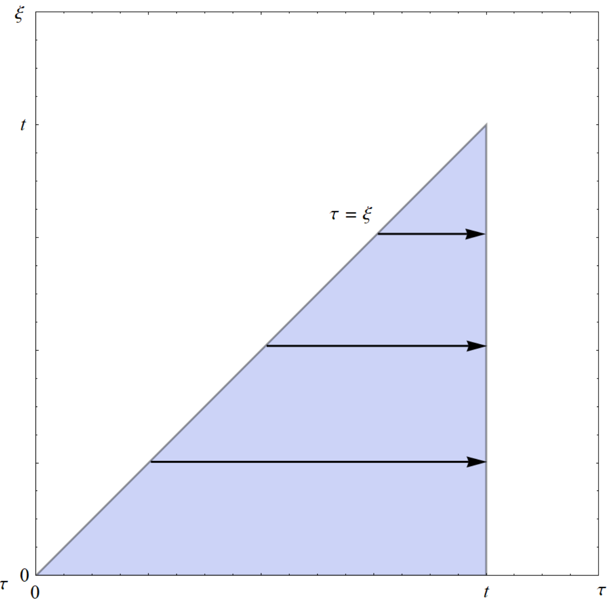
Part (c)

Start by using the definition of the convolution integral $f * (g * h)$.

$$\begin{aligned} f * (g * h) &= \int_0^t f(t-\tau)(g * h)(\tau)\,d\tau \\ &= \int_0^t f(t-\tau)\left[\int_0^\tau g(\tau-\xi)h(\xi)\,d\xi\right]d\tau \\ &= \int_0^t \int_0^\tau f(t-\tau)g(\tau-\xi)h(\xi)\,d\xi\,d\tau \end{aligned}$$

We want to make the substitution $u = \tau - \xi$ so that g will be in terms of a single variable. Doing this right now, though, will make h in terms of more than one variable because $d\xi$ comes first. Our aim then is to switch the order of integration to make τ come first. The current mode of integration in the $\tau\xi$-plane is shown below on the left.





Integrate over this domain as shown on the right to switch the order of integration.

$$f * (g * h) = \int_0^t \int_\xi^t f(t-\tau)g(\tau-\xi)h(\xi)\,d\tau\,d\xi$$

Now make the substitution $u = \tau - \xi$. Then $du = d\tau$.

$$\begin{aligned} f * (g * h) &= \int_0^t \int_0^{t-\xi} f(t-\xi-u)g(u)h(\xi)\,du\,d\xi \\ &= \int_0^t \left[\int_0^{t-\xi} f(t-\xi-u)g(u)\,du\right] h(\xi)\,d\xi \\ &= \int_0^t (f*g)(t-\xi)h(\xi)\,d\xi \\ &= (f*g)*h \end{aligned}$$